

SPECIAL IDENTITIES FOR THE PRE-JORDAN PRODUCT IN THE FREE DENDRIFORM ALGEBRA

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In memory of Jean-Louis Loday (1946–2012)

ABSTRACT. Pre-Jordan algebras were introduced recently in analogy with pre-Lie algebras. A pre-Jordan algebra is a vector space A with a bilinear multiplication $x \cdot y$ such that the product $x \circ y = x \cdot y + y \cdot x$ endows A with the structure of a Jordan algebra, and the left multiplications $L.(x): y \mapsto x \cdot y$ define a representation of this Jordan algebra on A . Equivalently, $x \cdot y$ satisfies these multilinear identities:

$$\begin{aligned} & (x \circ y) \cdot (z \cdot u) + (y \circ z) \cdot (x \cdot u) + (z \circ x) \cdot (y \cdot u) \\ & \equiv z \cdot [(x \circ y) \cdot u] + x \cdot [(y \circ z) \cdot u] + y \cdot [(z \circ x) \cdot u], \\ & x \cdot [y \cdot (z \cdot u)] + z \cdot [y \cdot (x \cdot u)] + [(x \circ z) \circ y] \cdot u \\ & \equiv z \cdot [(x \circ y) \cdot u] + x \cdot [(y \circ z) \cdot u] + y \cdot [(z \circ x) \cdot u]. \end{aligned}$$

The pre-Jordan product $x \cdot y = x \succ y + y \prec x$ in any dendriform algebra also satisfies these identities. We use computational linear algebra based on the representation theory of the symmetric group to show that every identity of degree ≤ 7 for this product is implied by the identities of degree 4, but that there exist new identities of degree 8 which do not follow from those of lower degree. There is an isomorphism of S_8 -modules between these new identities and the special identities for the Jordan diproduct in an associative dialgebra.

1. INTRODUCTION

During the last few decades many new algebraic structures have been discovered, connecting several areas in mathematics and physics. In particular, Loday [23] defined a dendriform algebra to be a vector space A with two bilinear operations \succ, \prec satisfying these identities:

$$\begin{aligned} & (x \prec y) \prec z \equiv x \prec (y \prec z) + x \prec (y \succ z), \\ & (x \succ y) \prec z \equiv x \succ (y \prec z), \\ & x \succ (y \succ z) \equiv (x \prec y) \succ z + (x \succ y) \succ z. \end{aligned}$$

The operation $x * y = x \prec y + x \succ y$ is associative, and so this new structure dichotomizes the notion of associative algebra, splitting the associative product into a sum of two operations. Dendriform algebras are also closely related to associative dialgebras, through the framework of Koszul duality for algebraic operads. The notion of associative dialgebra defines a binary quadratic operad \mathbf{Dias} , and the dual

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operad Dias^1 as defined by Ginzburg and Kapranov [17] is precisely the operad Dend of dendriform algebras. Further splitting of the operations gives rise to the quadri-algebras introduced by Aguiar and Loday [1]. For a general theory of splitting operations, see Bai et al. [2] and Vallette [27]. For an introduction to operads from the point of view of algebraic structures, see the lecture notes of Vatne [28]. For a comprehensive monograph on algebraic operads, see Loday and Vallette [25].

Pre-Lie and pre-Jordan algebras. The operation $x \cdot y = x \succ y - y \prec x$ in a dendriform algebra satisfies the defining identities for pre-Lie algebras, which had already been introduced independently in several areas of mathematics and then appeared in other fields; see [13] and the references therein. Pre-Lie algebras can be regarded as the algebraic structures behind the classical Yang-Baxter equation, which plays an important role in integrable systems and quantum groups. The L-dendriform algebras introduced by Bai, Liu and Ni [4] are related to pre-Lie algebras in the same way that pre-Lie algebras are related to Lie algebras.

We remark that the definition of dendriform algebra can be motivated in terms of pre-Lie algebras. Starting with the pre-Lie identity,

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) \equiv (y \cdot x) \cdot z - y \cdot (x \cdot z),$$

expanding it using the preceding operation in a dendriform algebra, and identifying terms with the same permutation of the variables gives these identities,

$$\begin{aligned} (x \succ y) \succ z - x \succ (y \succ z) &\equiv -(x \prec y) \succ z, \\ x \succ (z \prec y) &\equiv (x \succ z) \prec y, \\ -(y \prec x) \succ z &\equiv (y \succ x) \succ z - y \succ (x \succ z), \\ (y \succ z) \prec x &\equiv y \succ (z \prec x), \\ -z \prec (x \succ y) &\equiv z \prec (x \prec y) - (z \prec x) \prec y, \\ z \prec (y \prec x) - (z \prec y) \prec x &\equiv -z \prec (y \succ x), \end{aligned}$$

which are equivalent to the identities defining dendriform algebras.

Pre-Jordan algebras as Jordan analogues of pre-Lie algebras were introduced by Hou, Ni and Bai [19]. Explicitly,

- if (A, \cdot) is a pre-Lie (respectively pre-Jordan) algebra then the product $[x, y] = x \cdot y - y \cdot x$ (resp. $x \circ y = x \cdot y + y \cdot x$) defines a Lie (resp. Jordan) algebra,
- a vector space with a bilinear product (A, \cdot) is a pre-Lie (respectively pre-Jordan) algebra if and only if $(A, [-, -])$ (resp. (A, \circ)) defined above is a Lie (resp. Jordan) algebra and (L, A) is a representation of $(A, [-, -])$ (resp. (A, \circ)).

There are analogues of the classical Yang-Baxter equation in Jordan algebras [29] and pre-Jordan algebras [19]. The J-dendriform algebras introduced by Bai and Hou [3] are related to pre-Jordan algebras in the same way that pre-Jordan algebras are related to Jordan algebras.

These structures are interconnected as shown in the commutative diagram of categories in Table 1, which has been described in detail by Bai and Hou [3]. We are most interested in the vertical arrows, which use analogues of Lie and Jordan products on an algebra in one variety to endow the underlying vector space with the structure of an algebra in another variety.

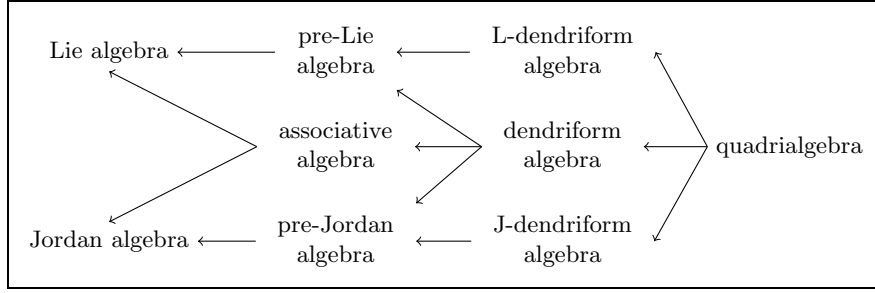


TABLE 1. Commutative diagram of categories of algebras

Pre-Lie algebras can be regarded as the algebraic structures behind the classical Yang-Baxter equation (CYBE), which plays an important role in integrable systems and quantum groups. This can be seen more clearly in terms of \mathcal{O} -operators of a Lie algebra introduced by Kupershmidt [21] who generalized the operator form of the CYBE in a Lie algebra. Moreover, the CYBE and the Rota-Baxter operators on a Lie algebra are understood as the special \mathcal{O} -operators corresponding to the co-adjoint representation and the adjoint representation. In terms of \mathcal{O} -operators, there are analogues of the CYBE in Jordan algebras [29] and pre-Jordan algebras [19]. The arrows in Table 1 can be reversed using Rota-Baxter operators [1, 3, 4]:

- a Rota-Baxter operator on an associative algebra gives a dendriform algebra,
- a Rota-Baxter operator on a dendriform algebra, or two commuting Rota-Baxter operators on an associative algebra, gives a quadri-algebra,
- a Rota-Baxter operator on a Lie algebra gives a pre-Lie algebra,
- a Rota-Baxter operator on a pre-Lie algebra, or two commuting Rota-Baxter operators on a Lie algebra, gives an L-dendriform algebra,
- a Rota-Baxter operator on a Jordan algebra gives a pre-Jordan algebra,
- a Rota-Baxter operator on a pre-Jordan algebra, or two commuting Rota-Baxter operators on a Jordan algebra, gives a J-dendriform algebra.

Polynomial identities for bilinear operations. In the present paper we investigate polynomial identities satisfied by the pre-Jordan product $x \cdot y = x \succ y + y \prec x$ in the free dendriform algebra; our approach relies heavily on computational linear algebra. We first verify that there are no identities of degree 3; we then consider degree 4 and show that the identities satisfied by this operation are equivalent to the defining identities for pre-Jordan algebras. We next show that every identity of degree ≤ 7 satisfied by this operation is implied by the identities of degree 4. We finally consider degree 8, and obtain special identities for the pre-Jordan product: identities satisfied by this operation but not implied by the identities of degree 4.

Comparing these results with those of Bremner and Peresi [12] for Jordan dialgebras, we observe some remarkable facts. The Jordan diproduct $x \dashv y + y \vdash x$ in an associative dialgebra satisfies the right commutative identity in degree 3; but the pre-Jordan product in a dendriform algebra satisfies no identities in degree 3. For each $n \geq 4$, we obtain two modules over the symmetric group S_n :

- The first S_n -module consists of the identities for the Jordan diproduct in degree n which do not follow from identities of lower degree, where an identity is a multilinear polynomial in the free right commutative algebra which expands to zero in the free associative dialgebra when the right commutative operation is replaced by the Jordan diproduct.
- The second S_n -module consists of the identities for the pre-Jordan product in degree n which do not follow from identities of lower degree, where an identity is a multilinear polynomial in the free nonassociative algebra which expands to zero in the free dendriform algebra when the nonassociative operation is replaced by the pre-Jordan product.

We show that these S_n -modules are isomorphic for $4 \leq n \leq 8$. This suggests a relation between the identities satisfied by corresponding bilinear operations in the free algebras in two varieties defined by dual operads. We give a precise statement of this conjecture; see Section 8.

Computational methods. We conclude this introduction with a brief summary of the theoretical aspects of our computational approach. For further information, see [10, 12, 18] and the references therein.

We say that a multilinear nonassociative polynomial $f(x_1, \dots, x_n)$ is an identity in an algebra A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$; we usually abbreviate this by $f(x_1, \dots, x_n) \equiv 0$. We regard the subspace of all identities of degree n for an algebra A as a module over the symmetric group S_n acting by permutations of the variables. Given identities f, f_1, \dots, f_k of degree n , we say that f is a consequence of f_1, \dots, f_k if f belongs to the S_n -submodule generated by f_1, \dots, f_k ; in other words, f is a linear combination of f_1, \dots, f_k , allowing permutations of the variables.

Let $\text{Non}(X)$ be the free nonassociative algebra on the generating set X over the field F , and let A_n be the subspace of $\text{Non}(X)$ consisting of the multilinear polynomials of degree n . Let $\text{Den}(X)$ be the free dendriform algebra on X over F , and let B_n be the subspace of $\text{Den}(X)$ consisting of the multilinear polynomials of degree n . For each n we consider the linear map $\mathcal{E}_n: A_n \rightarrow B_n$, which we call the expansion map, defined by replacing each occurrence of the nonassociative product $x \cdot y$ in $\text{Non}(X)$ by the pre-Jordan product $x \succ y + y \prec x$ in $\text{Den}(X)$. The basic computational principle is that the kernel of \mathcal{E}_n consists of the identities of degree n satisfied by the pre-Jordan product in the free dendriform algebra. As n grows, so does the size of the matrices, so different techniques are needed to compute this kernel, as will be detailed below.

All the computations described in this paper were performed using Maple 16. In general, we assume that the base field F has characteristic 0, and use rational arithmetic for our computations. However, as the sizes of the matrices increase, it becomes impossible to compute row canonical forms using rational arithmetic. In higher degrees, we therefore use modular arithmetic with a suitable prime p . To justify this, recall that the polynomial identities we consider are multilinear, so the spaces of identities in degree n have a natural structure of a module over the symmetric group S_n . The group algebra FS_n is semisimple when the base field F has characteristic 0 or $p > n$. This implies that the ranks of the matrices will be the same whether we use rational arithmetic or modular arithmetic with a prime greater than the degree of the identities. (We use $p = 101$, the smallest prime greater than 100.) For a more detailed discussion of these issues, see [12, §5].

2. NORMAL WORDS IN THE FREE DENDRIFORM ALGEBRA

An L -algebra is a vector space over a field F with two bilinear operations \prec, \succ satisfying inner associativity (also called the entanglement identity):

$$(1) \quad (x \succ y) \prec z \equiv x \succ (y \prec z).$$

Thus a dendriform algebra is an L -algebra satisfying two additional identities:

$$(2) \quad (x \prec y) \prec z \equiv x \prec (y \prec z) + x \prec (y \succ z),$$

$$(3) \quad x \succ (y \succ z) \equiv (x \succ y) \succ z + (x \prec y) \succ z.$$

The free dendriform algebra $\text{Den}(X)$ on a set X of generators is the quotient of the free L -algebra $L(X)$ by the T -ideal generated by these two identities.

We recall some definitions from Bokut et al. [5]. We consider a field F , a set of variables X , and a set of multilinear operations

$$\Omega = \bigcup_{n \geq 1} \Omega_n, \quad \Omega_n = \{\delta_i^{(n)} \mid i \in I_n\},$$

where Ω_n is the set of n -ary operations. We define the set of Ω -words over X as

$$(X, \Omega) = \bigcup_{m=0}^{\infty} (X, \Omega)_m, \quad (X, \Omega)_0 = X, \quad (X, \Omega)_m = X \cup \Omega((X, \Omega)_{m-1}) \quad (m \geq 1),$$

$$\Omega((X, \Omega)_{m-1}) = \bigcup_{t=1}^{\infty} \{\delta_i^{(t)}(u_1, \dots, u_t) \mid \delta_i^{(t)} \in \Omega_t, u_j \in (X, \Omega)_{m-1}\}.$$

Thus $(X, \Omega)_m$ consists of all monomials involving m or fewer operations.

If $\Omega = \{\succ, \prec\}$ then an Ω -word will be called an L -word. Chen and Wang [14] define an L -word u to be a normal L -word if u is one of the following:

- $u = x$, where $x \in X$.
- $u = v \succ w$, where v and w are normal L -words.
- $u = v \prec w$, with $v \neq v_1 \succ v_2$, where v, v_1, v_2, w are normal L -words.

The set N of normal L -words is a linear basis for the free L -algebra $L(X)$. Furthermore, there exists a monomial order in $L(X)$ which allows us to identify the leading normal L -word of any L -polynomial. Chen and Wang [14] prove that

$$\begin{aligned} S &= \{f_1(x, y, z), f_2(x, y, z), f_3(x, y, z, v) \mid x, y, z, v \in N\}, \\ f_1(x, y, z) &= (x \prec y) \prec z - x \prec (y \prec z) - x \prec (y \succ z), \\ f_2(x, y, z) &= (x \prec y) \succ z + (x \succ y) \succ z - x \succ (y \succ z), \\ f_3(x, y, z, v) &= ((x \succ y) \succ z) \succ v - (x \succ y) \succ (z \succ v) + (x \succ (y \prec z)) \succ v, \end{aligned}$$

is a Gröbner-Shirshov basis in $L(X)$ for the T -ideal generated by the dendriform identities (2)-(3).

Following Chen and Wang [14], we say that an L -word u is a normal D -word if u is one of the following:

- $u = x$, where $x \in X$.
- $u = x \prec v$, where $x \in X$ and v is a normal D -word.
- $u = x \succ v$, where $x \in X$ and v is a normal D -word.
- $u = (x \succ u_1) \succ u_2$, where $x \in X$ and u_1, u_2 are normal D -words.

The set of normal D -words is a linear basis of the free dendriform algebra [14, Corollary 3.5]. This allows us to reduce substantially the size of the computations in the free dendriform algebra, as we can see from Table 2. The numbers of binary association types (one nonassociative binary operation), double binary association types (two nonassociative binary operations), and normal D -types, are given by

$$\frac{1}{n} \binom{2n-2}{n-1}, \quad 2^{n-1} \frac{1}{n} \binom{2n-2}{n-1}, \quad \frac{1}{n+1} \binom{2n}{n}.$$

The first formula is well-known, and the second is an immediate consequence; for the third see Loday [23, §5 and Appendix A].

n	binary types	double binary types	normal D -types
3	2	8	5
4	5	40	14
5	14	224	42
6	42	1344	132
7	132	8448	429
8	429	54912	1430

TABLE 2. Binary association types and normal dendriform words

Figure 1 gives an algorithm to express any L -word as a linear combination of normal D -words using the reductions given by the Gröbner-Shirshov basis.

3. NON-EXISTENCE OF IDENTITIES IN DEGREE 3

Since the defining identities for dendriform algebras have degree 3, we need to check whether the pre-Jordan product in the free dendriform algebra satisfies any multilinear identities in degree 3. In case of a positive answer, it would be natural to include these identities in the definition of pre-Jordan algebra.

Since the pre-Jordan product satisfies no identities in degree 2, we need to consider both possible association types in degree 3 for a binary nonassociative product: $(- \cdot -) \cdot -$ and $- \cdot (- \cdot -)$. Applying 6 permutations of 3 variables x, y, z to the 2 association types gives 12 multilinear nonassociative monomials:

$$(x \cdot y) \cdot z, (x \cdot z) \cdot y, (y \cdot x) \cdot z, (y \cdot z) \cdot x, (z \cdot x) \cdot y, (z \cdot y) \cdot x, \\ x \cdot (y \cdot z), x \cdot (z \cdot y), y \cdot (x \cdot z), y \cdot (z \cdot x), z \cdot (x \cdot y), z \cdot (y \cdot x).$$

These 12 monomials form a basis of the space A_3 of multilinear nonassociative polynomials of degree 3.

In degree 3 there are 8 association types for an algebra with 2 binary nonassociative products:

$$(- \prec -) \prec -, (- \prec -) \succ -, (- \succ -) \prec -, (- \succ -) \succ -, \\ - \prec (- \prec -), - \succ (- \prec -), - \prec (- \succ -), - \succ (- \succ -).$$

The Gröbner-Shirshov basis for the free dendriform algebra [14] allows us to reduce this set to the 5 types, which we call normal D -types:

$$- \prec (- \prec -), - \succ (- \prec -), - \prec (- \succ -), - \succ (- \succ -), (- \succ -) \succ -.$$

Procedure **Dnormalize**

Input: a term $[c, m]$ consisting of a coefficient c and an L -word m .

Output: a list containing the terms of the normal form of $[c, m]$ as a linear combination of normal D -words.

- If the degree of m is 1 or 2 then
 - Set **checkedresult** $\leftarrow [c, m]$.
- Else (we write $m = m_1 \circ m_2$ where $\circ \in \{\succ, \prec\}$)
 - Set **result** $\leftarrow []$ (the empty list).
 - Set **normalized1** $\leftarrow \mathbf{Dnormalize}([1, m_1])$ (recursive call).
 - Set **normalized2** $\leftarrow \mathbf{Dnormalize}([1, m_2])$ (recursive call).
 - For $[c_1, m'_1]$ in **normalized1** and $[c_2, m'_2]$ in **normalized2** do:
 - * Set **newc** $\leftarrow cc_1c_2$.
 - * Set **newm** $\leftarrow m'_1 \circ m'_2$.
 - * If **newm** has the form $(x \succ y) \prec z$ then:
 - Append the term $[\mathbf{newc}, x \succ (y \prec z)]$ to **result** (reduce using inner associativity).
 - * Else if **newm** has the form $(x \prec y) \prec z$ then:
 - Append $[\mathbf{newc}, x \prec (y \prec z)], [\mathbf{newc}, x \prec (y \succ z)]$ to **result** (reduce using f_1).
 - * Else if **newm** has the form $(x \prec y) \succ z$ then:
 - Append $[-\mathbf{newc}, (x \succ y) \succ z], [\mathbf{newc}, x \succ (y \succ z)]$ to **result** (reduce using f_2).
 - * Else if **newm** has the form $((x \succ y) \succ z) \succ v$ then:
 - Append $[\mathbf{newc}, (x \succ y) \succ (z \succ v)], [-\mathbf{newc}, (x \succ (y \prec z)) \succ v]$ to **result** (reduce using f_3).
 - * Else append $[\mathbf{newc}, \mathbf{newm}]$ to **result**.
 - If **result** $= [[c, m]]$ then (no reduction was possible)
 - * Set **checkedresult** $\leftarrow \mathbf{result}$.
 - Else
 - * Set **checkedresult** $\leftarrow []$.
 - * While **result** $\neq []$ do:
 - Set **element** $\leftarrow \mathbf{result}[1]$ (the first term in **result**).
 - Set **normalizedelement** $\leftarrow \mathbf{Dnormalize}(\mathbf{element})$.
 - If **element** $= \mathbf{normalizedelement}[1]$ then:
 - Append **element** to **checkedresult**.
 - Delete **element** from **result**.
 - Else in **result** replace **element** by **normalizedelement**.
- Return **checkedresult**.

FIGURE 1. Algorithm to compute the normal form of an L -word

In degree 3 the difference between using all types and using normal D -types is not substantial, but as the degree becomes larger, the difference becomes bigger, and reductions must be done whenever possible to save computer memory. Applying 6 permutations of 3 variables to 5 normal D -types gives 30 multilinear normal D -words, which form a basis of the space B_3 of multilinear dendriform monomials of degree 3.

$$\begin{bmatrix}
. & . & . & . & . & 1 & . & . & . & . & . & 1 \\
. & . & . & 1 & . & . & . & . & . & 1 & . & . \\
. & . & . & . & 1 & . & . & . & . & . & 1 & . \\
. & 1 & . & . & . & . & . & 1 & . & . & . & . \\
. & . & 1 & . & . & . & . & . & 1 & . & . & . \\
1 & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & 1 & . & . \\
. & . & . & . & . & . & . & . & 1 & . & 1 & . \\
. & . & . & . & . & . & . & 1 & . & 1 & . & . \\
. & . & . & . & . & . & . & . & 1 & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1 & . & 1 \\
. & . & . & 1 & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & 1 & . & . & . & 1 & . & . \\
. & 1 & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & 1 & . & . & 1 & . & . & . & . \\
1 & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & 1 & . & . & . & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & . & . & 1 & . & . & . & . \\
1 & . & . & . & . & . & . & . & 1 & . & . & . \\
. & 1 & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & 1 & . & . & . & . & . & . & 1 & . \\
1 & . & -1 & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & -1 & . & . & . & . & . & . & . \\
-1 & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & -1 & . & . & . & . & . & . \\
. & -1 & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & -1 & . & 1 & . & . & . & . & . & .
\end{bmatrix}$$

TABLE 3. Expansion matrix in degree 3

Proposition 3.1. *The pre-Jordan product in the free dendriform algebra satisfies no multilinear identities in degree 3.*

Proof. The identities in degree 3 are the nonzero elements of the kernel of the expansion map $\mathcal{E}_3: A_3 \rightarrow B_3$. The 30×12 expansion matrix E_3 represents this linear map with respect to the monomial bases of A_3 and B_3 described above. The (i, j) entry of E_3 contains the coefficient of the i -th normal D -word in the normalized expansion of the j -th pre-Jordan monomial; to normalize the terms of the expansion, we apply the algorithm in Figure 1.

We consider the basic monomial in each association type, by which we mean the monomial with the identity permutation of the variables. For both association types, the expansion has 4 terms, but after normalization we obtain 5 terms:

$$\begin{aligned}
(x \cdot y) \cdot z &= (x \succ y) \succ z + z \prec (x \succ y) + (y \prec x) \succ z + z \prec (y \prec x) \\
&= (x \succ y) \succ z + z \prec (x \succ y) - (y \succ x) \succ z + y \succ (x \succ z) + z \prec (y \prec x), \\
x \cdot (y \cdot z) &= x \succ (y \succ z) + (y \succ z) \prec x + x \succ (z \prec y) + (z \prec y) \prec x \\
&= x \succ (y \succ z) + (y \succ z) \prec x + x \succ (z \prec y) + z \prec (y \prec x) + z \prec (y \succ x).
\end{aligned}$$

The expansions of the other basis monomials for A_3 are obtained by permutation of the variables.

The resulting expansion matrix is displayed in Table 3. This matrix has full rank, and hence its nullspace is $\{0\}$. \square

Remark 3.2. Throughout this paper we will be comparing the identities satisfied by the pre-Jordan product in the free dendriform algebra with the identities satisfied by the Jordan diproduct in the free associative dialgebra [12]. Every multilinear identity of degree 3 satisfied by the Jordan diproduct in the free associative dialgebra is a consequence of the right commutative identity $x(yz) \equiv x(z y)$ [7, Theorem 6]. This explains why in [12] the domain of the expansion map in higher degrees was taken to be the multilinear subspace of the free right commutative algebra. However, in this paper, since we have no identities in degree 3, the domain of the expansion map will be the multilinear subspace of the free nonassociative algebra.

4. EXISTENCE OF DEFINING IDENTITIES IN DEGREE 4

To find the multilinear polynomial identities in degree 4 satisfied by the pre-Jordan product in the free dendriform algebra, we proceed as in degree 3, but the expansion matrix E_4 is larger. We show that the defining identities for pre-Jordan algebras generate the kernel of the expansion map \mathcal{E}_4 as an S_4 -module.

In degree 4 there are 5 association types for a nonassociative binary product:

$$((-\cdot-\cdot)\cdot-\cdot), \quad (-\cdot(-\cdot-\cdot))\cdot-, \quad (-\cdot-\cdot)\cdot(-\cdot-\cdot), \quad -\cdot((-\cdot-\cdot)\cdot-), \quad -\cdot(-\cdot(-\cdot-\cdot)).$$

Applying 24 permutations of 4 variables x, y, z, v we get 120 multilinear monomials which form a basis of A_4 . We order these monomials first by association type and then by lexicographical order of the permutation.

In an algebra with 2 nonassociative binary operations, there are 40 association types, which reduce to 14 normal D -types:

$$(4) \quad \left\{ \begin{array}{lll} - \prec (- \prec (- \prec -)), & - \succ (- \prec (- \prec -)), & - \prec (- \succ (- \prec -)), \\ - \succ (- \succ (- \prec -)), & - \prec (- \prec (- \succ -)), & - \succ (- \prec (- \succ -)), \\ - \prec (- \succ (- \succ -)), & - \succ (- \succ (- \succ -)), & - \prec ((- \succ -) \succ -), \\ - \succ ((- \succ -) \succ -), & (- \succ -) \succ (- \prec -), & (- \succ -) \succ (- \succ -), \\ (- \succ (- \prec -)) \succ -, & (- \succ (- \succ -)) \succ -. \end{array} \right.$$

Applying permutations of the variables we get 336 multilinear normal D -words which form a basis for the space B_4 . We order these normal D -words first by D -type and then by lex order of the permutation.

Theorem 4.1. *The kernel of the expansion map $\mathcal{E}_4: A_4 \rightarrow B_4$ is generated as an S_4 -module by the defining identities for pre-Jordan algebras:*

$$\begin{aligned} PJ_1(x, y, z, u) &= (x \circ y) \cdot (z \cdot u) + (y \circ z) \cdot (x \cdot u) + (z \circ x) \cdot (y \cdot u) \\ &\quad - z \cdot [(x \circ y) \cdot u] - x \cdot [(y \circ z) \cdot u] - y \cdot [(z \circ x) \cdot u] \equiv 0, \\ PJ_2(x, y, z, u) &= x \cdot [y \cdot (z \cdot u)] + z \cdot [y \cdot (x \cdot u)] + [(x \circ z) \circ y] \cdot u \\ &\quad - z \cdot [(x \circ y) \cdot u] - x \cdot [(y \circ z) \cdot u] - y \cdot [(z \circ x) \cdot u] \equiv 0, \end{aligned}$$

where $x \circ y = x \cdot y + y \cdot x$.

Proof. The 336×120 matrix E_4 represents the expansion map $\mathcal{E}_4: A_4 \rightarrow B_4$ with respect to the given ordered bases of A_4 and B_4 . The (i, j) entry of E_4 is the coefficient of the i -th normal D -word in the normalized expansion of the j -th nonassociative monomial. We display the expansions of the 5 association types with the identity

permutation of the variables. Each expansion produces 8 terms, but normalization increases the number to 15, 13, 12, 14, 16 terms respectively:

$$\begin{aligned}
((x \cdot y) \cdot z) \cdot u &= ((x \succ y) \succ z) \succ u + u \prec ((x \succ y) \succ z) + (z \prec (x \succ y)) \succ u \\
&\quad + u \prec (z \prec (x \succ y)) + ((y \prec x) \succ z) \succ u + u \prec ((y \prec x) \succ z) \\
&\quad + (z \prec (y \prec x)) \succ u + u \prec (z \prec (y \prec x)) \\
&= (x \succ y) \succ (z \succ u) - (x \succ (y \prec z)) \succ u + u \prec ((x \succ y) \succ z) \\
&\quad - (z \succ (x \succ y)) \succ u + z \succ ((x \succ y) \succ u) + u \prec (z \prec (x \succ y)) \\
&\quad - (y \succ x) \succ (z \succ u) + (y \succ (x \prec z)) \succ u + (y \succ (x \succ z)) \succ u \\
&\quad - u \prec ((y \succ x) \succ z) + u \prec (y \succ (x \succ z)) - (z \succ (y \prec x)) \succ u \\
&\quad - z \succ ((y \succ x) \succ u) + z \succ (y \succ (x \succ u)) + u \prec (z \prec (y \prec x)), \\
(x \cdot (y \cdot z)) \cdot u &= (x \succ (y \succ z)) \succ u + u \prec (x \succ (y \succ z)) + ((y \succ z) \prec x) \succ u \\
&\quad + u \prec ((y \succ z) \prec x) + (x \succ (z \prec y)) \succ u + u \prec (x \succ (z \prec y)) \\
&\quad + ((z \prec y) \prec x) \succ u + u \prec ((z \prec y) \prec x) \\
&= (x \succ (y \succ z)) \succ u + u \prec (x \succ (y \succ z)) + (y \succ (z \prec x)) \succ u \\
&\quad + u \prec (y \succ (z \prec x)) + (x \succ (z \prec y)) \succ u + u \prec (x \succ (z \prec y)) \\
&\quad - (z \succ (y \prec x)) \succ u - z \succ ((y \succ x) \succ u) + z \succ (y \succ (x \succ u)) \\
&\quad - (z \succ (y \succ x)) \succ u + z \succ ((y \succ x) \succ u) + u \prec (z \prec (y \prec x)) \\
&\quad + u \prec (z \prec (y \succ x)), \\
(x \cdot y) \cdot (z \cdot u) &= (x \succ y) \succ (z \succ u) + (z \succ u) \prec (x \succ y) + (x \succ y) \succ (u \prec z) \\
&\quad + (u \prec z) \prec (x \succ y) + (y \prec x) \succ (z \succ u) + (z \succ u) \prec (y \prec x) \\
&\quad + (y \prec x) \succ (u \prec z) + (u \prec z) \prec (y \prec x) \\
&= (x \succ y) \succ (z \succ u) + z \succ (u \prec (x \succ y)) + (x \succ y) \succ (u \prec z) \\
&\quad + u \prec (z \prec (x \succ y)) + u \prec (z \succ (x \succ y)) - (y \succ x) \succ (z \succ u) \\
&\quad + y \succ (x \succ (z \succ u)) + z \succ (u \prec (y \prec x)) - (y \succ x) \succ (u \prec z) \\
&\quad + y \succ (x \succ (u \prec z)) + u \prec (z \prec (y \prec x)) + u \prec (z \succ (y \prec x)), \\
x \cdot ((y \cdot z) \cdot u) &= x \succ ((y \succ z) \succ u) + ((y \succ z) \succ u) \prec x + x \succ (u \prec (y \succ z)) \\
&\quad + (u \prec (y \succ z)) \prec x + x \succ ((z \prec y) \succ u) + ((z \prec y) \succ u) \prec x \\
&\quad + x \succ (u \prec (z \prec y)) + (u \prec (z \prec y)) \prec x \\
&= x \succ ((y \succ z) \succ u) + (y \succ z) \succ (u \prec x) + x \succ (u \prec (y \succ z)) \\
&\quad + u \prec (y \succ (z \prec x)) + u \prec ((y \succ z) \succ x) - x \succ ((z \succ y) \succ u) \\
&\quad + x \succ (z \succ (y \succ u)) - (z \succ y) \succ (u \prec x) + z \succ (y \succ (u \prec x)) \\
&\quad + x \succ (u \prec (z \prec y)) + u \prec (z \prec (y \prec x)) + u \prec (z \prec (y \succ x)) \\
&\quad - u \prec ((z \succ y) \succ x) + u \prec (z \succ (y \succ x)), \\
x \cdot (y \cdot (z \cdot u)) &= x \succ (y \succ (z \succ u)) + (y \succ (z \succ u)) \prec x + x \succ ((z \succ u) \prec y) \\
&\quad + ((z \succ u) \prec y) \prec x + x \succ (y \succ (u \prec z)) + (y \succ (u \prec z)) \prec x \\
&\quad + x \succ ((u \prec z) \prec y) + ((u \prec z) \prec y) \prec x \\
&= x \succ (y \succ (z \succ u)) + y \succ (z \succ (u \prec x)) + x \succ (z \succ (u \prec y)) \\
&\quad + z \succ (u \prec (y \prec x)) + z \succ (u \prec (y \succ x)) + x \succ (y \succ (u \prec z)) \\
&\quad + y \succ (u \prec (z \prec x)) + y \succ (u \prec (z \succ x)) + x \succ (u \prec (z \prec y)) \\
&\quad + x \succ (u \prec (z \succ y)) + u \prec (z \prec (y \prec x)) + u \prec (z \prec (y \succ x)) \\
&\quad - u \prec ((z \succ y) \succ x) + u \prec (z \succ (y \succ x)) + u \prec (z \succ (y \prec x))
\end{aligned}$$

$$+ u \prec ((z \succ y) \succ x).$$

We compute the rank of the expansion matrix and obtain 104, so the nullity is 16, showing that there are identities in degree 4.

The most natural way to obtain an integer basis for the nullspace of an integer matrix is to use the Hermite normal form. In the present case, this produces an invertible 120×120 integer matrix U for which $UE_4^t = H$ where H is the Hermite normal form of E_4^t . Since E_4 has rank 104, the last 16 rows of H are zero, and so the last 16 rows of U form a basis for the left nullspace of E_4^t which is the right nullspace of E_4 . The squared Euclidean lengths of these nullspace basis vectors are 12 (4 times), 24 (6 times), 36 (3 times), 48 (3 times).

To get shorter vectors, we apply the LLL algorithm for lattice basis reduction [8] to these nullspace basis vectors. This gives 16 vectors all with squared Euclidean length 12, which we denote by V_1, \dots, V_{16} . For further details on this method as applied to polynomial identities, see [11].

We compare the resulting identities (whose coefficient vectors are the nullspace basis vectors) with the defining identities for pre-Jordan algebras to verify that they are equivalent; that is, that they generate the same S_4 -module. We construct a matrix with 144 rows and 120 columns, consisting of a 120×120 upper block and a 24×120 lower block. For each of the 2 defining identities for pre-Jordan algebras, we fill the lower block with the coefficient vectors obtained by applying all 24 permutations of the variables and then compute the row canonical form (RCF). After processing both identities, the rank is 16. Retaining the results of this computation, we perform the same calculation for each of the 16 nullspace basis identities described above. The rank does not increase, and hence the nullspace basis identities are consequences of the defining identities for pre-Jordan algebras.

We then reverse this calculation, first processing the nullspace basis identities, obtaining rank 16, and then processing the defining identities, which do not increase the rank. Hence the defining identities are consequences of the nullspace basis identities. This completes the proof. \square

S_4 -module structure of identities. As mentioned in Remark 3.2, we will compare the identities for the pre-Jordan product in the free dendriform algebra with those for the Jordan diproduct in the free associative dialgebra [12]. In degree 4, both spaces are nonzero, so we can explicitly compute the structure of the identities as modules over the symmetric group S_4 . Table 4 gives the character table of S_4 .

partition	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
4	1	1	1	1	1
31	3	1	-1	0	-1
22	2	0	2	-1	0
211	3	-1	-1	0	1
1111	1	-1	1	1	-1

TABLE 4. Character table of the symmetric group S_4

We write N_4 for the nullspace of the expansion matrix E_4 . From the last proof we know that the reduced vectors V_1, \dots, V_{16} form a basis for N_4 . Each of these vectors has squared length 12, and has 12 nonzero components from $\{\pm 1\}$.

To compute the character of N_4 as an S_4 -module, we choose a set of conjugacy class representatives (the column labels in Table 4), and compute the matrix representing the action of each representative on N_4 with respect to the basis V_1, \dots, V_{16} . The traces of these matrices give the character $(16, 4, 0, 1, 0)$ of the S_4 -module N_4 . Expressing this character as a linear combination of the rows of the character table gives the decomposition of N_4 as a sum of irreducible S_4 -submodules:

$$(5) \quad N_4 \cong 2[4] \oplus 3[31] \oplus [22] \oplus [211],$$

where $m[\lambda]$ denotes m copies of the irreducible module for partition λ . We note that there are no copies of the signature module $[1111]$.

For the Jordan diproduct in the free associative dialgebra, the domain A_4 of the expansion map \mathcal{E}_4 is the multilinear subspace in degree 4 of the free right commutative algebra, and B_4 is the multilinear subspace in degree 4 of the free associative dialgebra. A basis for the kernel of this expansion map is given by the 16 rows of the lower block in the matrix of [7, Table 4]. Following the methods of the previous paragraph, we determine that the nullspace for the Jordan diproduct has the same character as the nullspace for the pre-Jordan product.

Proposition 4.2. *In degree 4, the S_4 -module of multilinear nonassociative polynomial identities satisfied by the pre-Jordan product in the free dendriform algebra is isomorphic to the S_4 -module of multilinear right-commutative polynomial identities satisfied by the Jordan diproduct in the free associative dialgebra.*

As we will see throughout this paper, this fact is not accidental: in each degree $4 \leq n \leq 8$, there is an S_n -module isomorphism between the new identities satisfied by the pre-Jordan product in the free dendriform algebra and the new identities satisfied by the quasi-Jordan product in the free associative dialgebra. (An identity is called new if it does not follow from identities of lower degree.)

5. REPRESENTATION THEORY OF THE SYMMETRIC GROUP

Since the sizes of the matrices we use are growing, it is useful to introduce another method to compute and compare identities. This new method is more sophisticated and involves deeper knowledge of the representation theory of the symmetric group. The basic idea consists in breaking down a module of identities into its irreducible components and doing the computations in these smaller submodules. Furthermore, using representation theory allows us to consider only the basic monomial in each association type (the monomial with the identity permutation of the variables); without representation theory, we have to use all permutations of the variables, which significantly increases the size of the computations.

Theoretical background. Given $f(x_1, \dots, x_n)$, a multilinear nonassociative polynomial of degree n , which we regard as a polynomial identity for an algebra, we collect its terms by association type and write $f = f_1 + \dots + f_t$, where t is the number of association types. In each association type a term can be identified by its coefficient $c \in F$ and the permutation π of the variables in the monomial. Thus each f_i can be expressed as an element

$$g_i = \sum_{\pi \in S_n} c_{i\pi} \pi,$$

of the group algebra FS_n , and hence f can be identified with the element (g_1, \dots, g_t) of $M = FS_n \oplus \dots \oplus FS_n$ (t summands). If $\pi \in S_n$ then $(\pi g_1, \dots, \pi g_t)$ is also

an identity, which represents the identity f applied to a permutation of its arguments. Since linear combinations of identities are also identities, it follows that (gg_1, \dots, gg_t) is an identity for any element $g \in FS_n$. Thus M is a module for the group algebra FS_n and the set of identities which are consequences of f is a submodule of M .

The partitions λ of n are in bijection with the isomorphism classes of irreducible representations of S_n ; we write d_λ for the dimension of the irreducible representation corresponding to λ . By Wedderburn's theorem, in the case $F = \mathbb{Q}$, the group algebra $\mathbb{Q}S_n$ is isomorphic to the direct sum of matrix algebras of size $d_\lambda \times d_\lambda$:

$$\rho: \mathbb{Q}S_n \xrightarrow{\cong} \bigoplus_{\lambda} M_{d_\lambda}(\mathbb{Q}).$$

If ρ_λ is the projection of the group algebra onto the matrix subalgebra corresponding to the partition λ , then we associate to each element of the group algebra a matrix of size $d_\lambda \times d_\lambda$. Thus $[\rho_\lambda(g_1) \cdots \rho_\lambda(g_t)]$ is a matrix of size $d_\lambda \times td_\lambda$ which corresponds to the identity f in the representation associated to λ . In the same way,

$$[\rho_\lambda(gg_1) \cdots \rho_\lambda(gg_t)] = \rho_\lambda(g)[\rho_\lambda(g_1) \cdots \rho_\lambda(g_t)]$$

corresponds to a sequence of row operations applied to the matrix representing f . (We are primarily interested in computing the RCFs of these matrices, which includes the possibility of a nonzero row being reduced to zero.) The matrices $\rho_\lambda(g)$ are computed using the algorithm of Clifton [15]; see also [12, Figure 1].

Two identities are equivalent if and only if they generate the same submodule of M . In other words, for each irreducible representation λ the matrices representing the two identities have the same row space. Stacking vertically the matrices corresponding to k identities $f^{(1)}, \dots, f^{(k)}$ gives a matrix of size $kd_\lambda \times td_\lambda$. Each row of this matrix represents an identity implied by $f^{(1)}, \dots, f^{(k)}$. Row operations on this matrix replace rows with linear combination of rows. Thus the nonzero rows of the RCF are a set of linearly independent generators for the submodule of M generated by $f^{(1)}, \dots, f^{(k)}$ corresponding to partition λ . Each of these nonzero rows represents a copy of the irreducible module $[\lambda]$ in the decomposition of this submodule.

Comparison with other methods. The method used in [12] (see Table 3) could be used to find all the polynomial identities in degree n satisfied by the pre-Jordan product in the free dendriform algebra, as follows. Let t be the number of binary nonassociative types in degree n , and let s be the number of normal D -types in degree n . Let λ be a partition of n with irreducible representation of dimension d_λ . We write X_λ for the matrix of size $td_\lambda \times sd_\lambda$ consisting of $d_\lambda \times d_\lambda$ blocks. To fill this matrix, we first compute the normalized expansions of the basic nonassociative monomials in the t association types. For each $i = 1, \dots, t$ and $j = 1, \dots, s$ we collect the terms in the expansion of the i -th basic monomial which have the j -th normal D -type. In the matrix X_λ , the block in position (i, j) is the representation matrix for the group algebra element corresponding to the j -th normal D -type in the i -th expansion. We now form the augmented matrix $[X_\lambda | I]$ where I is the identity matrix of order td_λ . Assume that X_λ has rank r . We compute the RCF of $[X_\lambda | I]$ to obtain $[R | U]$ where R is the RCF of X_λ and U is an invertible matrix for which $UX_\lambda = R$. The last $td_\lambda - r$ rows of U represent the identities in degree n and partition λ satisfied by the pre-Jordan product in the free dendriform algebra. Then we also have $X_\lambda = U^{-1}R$. Since the last $td_\lambda - r$ rows of R are zero, the

first r columns of U^{-1} show how to express the rows of X_λ in terms of the nonzero rows of R , and since the nonzero rows of R are linearly independent, the first r columns of U^{-1} are uniquely determined. Hence the first r rows of U are uniquely determined. Moreover, the last $td_\lambda - r$ rows of U are uniquely determined since they form a submatrix in RCF (these rows are the lower right block which has the identities we are looking for). Since $UX_\lambda = R$, it follows that rows $r+1$ to td_λ of U form a (canonical) basis of the left nullspace of X_λ , which coincides with the right nullspace of X_λ^t . So if we compute the RCF of X_λ^t , find the canonical basis of its nullspace, put these basis vectors into another matrix, and then compute its RCF, we will obtain the same matrix as the last $td_\lambda - r$ rows of U . We can therefore obtain the same results using X_λ^t , which in general is much smaller than $[X_\lambda|I]$.

Example: degree 4. We consider again the identities in degree 4 for the pre-Jordan product in the free dendriform algebra.

There are 5 partitions of 4, namely 4, 31, 22, 211, 1111; the corresponding irreducible representations of S_4 have dimensions 1, 3, 2, 3, 1 respectively. For each partition λ with corresponding dimension d_λ , we construct a matrix X_4 of size $5d_\lambda \times 14d_\lambda$. This matrix consists of 5 rows and 14 columns of $d_\lambda \times d_\lambda$ blocks. To fill this matrix, we first compute the normalized expansions of the basic nonassociative monomials in the 5 association types. For each $i = 1, \dots, 5$ and $j = 1, \dots, 14$ we collect the terms in the expansion of the i -th basic monomial which have the j -th normal D -type. For example, for $i = 1$ the sorted expansion of $((x \cdot y) \cdot z) \cdot u$ is

$$\begin{aligned} & [uzyx]_1 + [uzxy]_5 + [uyxz]_7 + [zyxu]_8 + [uxyz - uyxz]_9 + [zxyu - zyxu]_{10} \\ & + [xyzu - yxzu]_{12} + [-xyzu + yxzu - zyxu]_{13} + [-zxyu + yxzu]_{14}. \end{aligned}$$

The subscripts indicate the normal D -type in the order given in (4); within each pair of brackets, we give the corresponding element of the group algebra $\mathbb{Q}S_4$. In the matrix X_4 , the block in position (i, j) is the representation matrix for the group algebra element corresponding to the j -th normal D -type in the i -th expansion. The nullspace of the transpose matrix X_4^t represents the identities for the pre-Jordan corresponding to representation $[\lambda]$.

In particular, we consider $\lambda = 4$ with $d_\lambda = 1$. Isomorphism (5) shows that the representation [4] has multiplicity 2 in the S_4 -module of identities in degree 4 for the pre-Jordan product. Thus we expect that the nullspace of X_4^t will be 2-dimensional. We calculate X_4 ,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 3 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and obtain the row canonical form of X_4^t (omitting zero rows),

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix},$$

showing that the nullity is 2.

Similar calculations with $\lambda = 31, 22, 211, 1111$ produce nullities 3, 1, 1, 0 respectively, agreeing with the isomorphism (5).

6. NON-EXISTENCE OF SPECIAL IDENTITIES IN DEGREES 5, 6 AND 7

The pre-Jordan product satisfies the defining identities for pre-Jordan algebras in degree 4. It follows that the pre-Jordan product satisfies identities in every degree $n \geq 4$, which we call the liftings of the defining identities.

Definition 6.1. From a multilinear polynomial identity $f(x_1, x_2, \dots, x_n)$ in degree n for a nonassociative algebra with one binary operation $x \cdot y$, we obtain $n + 2$ identities in degree $n + 1$ which generate all the consequences of f in degree $n + 1$ as a module over the symmetric group S_{n+1} . We introduce a new variable x_{n+1} and perform n substitutions and 2 multiplications, obtaining the **liftings** of f to degree $n + 1$:

$$\begin{aligned} &f(x_1 \cdot x_{n+1}, x_2, \dots, x_n), \quad f(x_1, x_2 \cdot x_{n+1}, \dots, x_n), \quad \dots, \quad f(x_1, x_2, \dots, x_n \cdot x_{n+1}), \\ &f(x_1, x_2, \dots, x_n) \cdot x_{n+1}, \quad x_{n+1} \cdot f(x_1, x_2, \dots, x_n). \end{aligned}$$

Definition 6.2. Let $\Omega = \{\omega_1, \dots, \omega_\ell\}$ be operation symbols of arbitrary arities, and let \mathcal{V} be the variety of all Ω -algebras. Let \mathcal{W} be a variety of algebras defined by multilinear identities. Let $P = \{p_1, \dots, p_\ell\}$ be multilinear polynomials in \mathcal{W} such that the degree of p_i equals the arity of ω_i . The **expansion map** in degree n has domain equal to the multilinear subspace in degree n of the free \mathcal{V} -algebra and codomain equal to the multilinear subspace in degree n of the free \mathcal{W} -algebra; it is defined by replacing every occurrence of ω_i by p_i . We choose a degree m and consider the kernels of the expansion maps in all degrees $n \leq m$. We write f_1, \dots, f_k for a set of S_n -module generators of these kernels. Any element of the kernel of the expansion map in degree $n' > m$ which is not a consequence of the liftings of f_1, \dots, f_k to degree n' will be called a **special identity** for the operations P in every \mathcal{W} -algebra.

In our case, $\Omega = \{x \cdot y\}$ consists of a single bilinear operation, \mathcal{W} is the variety of dendriform algebras, $P = \{x \succ y + y \prec x\}$ consists of the pre-Jordan product in the free dendriform algebra, and $m = 4$ (the degree of the defining identities for pre-Jordan algebras).

Degree 5. We first need to lift the defining identities PJ_1, PJ_2 for pre-Jordan algebras (Theorem 4.1) to degree 5. For each $i = 1, 2$ we perform 4 substitutions and 2 multiplications, obtaining 12 identities in degree 5:

$$(6) \quad \begin{cases} PJ_i(x \cdot v, y, z, u), & PJ_i(x, y \cdot v, z, u), & PJ_i(x, y, z \cdot v, u), \\ PJ_i(x, y, z, u \cdot v), & PJ_i(x, y, z, u) \cdot v, & v \cdot PJ_i(x, y, z, u). \end{cases}$$

In degree 5 we have $t = 14$ association types for the binary nonassociative product. In the free dendriform algebra we have $s = 42$ normal D -types. There are 7 distinct irreducible representations of the symmetric group S_5 , with dimensions $d_\lambda = 1, 4, 5, 6, 5, 4, 1$ corresponding to partitions $\lambda = 5, 41, 32, 311, 221, 2111, 11111$. The representation matrices $\rho_i(\pi)$ ($1 \leq i \leq 7$) are given by the projections onto the corresponding simple ideals in the direct sum decomposition of the group algebra:

$$\mathbb{Q}S_5 = \mathbb{Q} \oplus M_4(\mathbb{Q}) \oplus M_5(\mathbb{Q}) \oplus M_6(\mathbb{Q}) \oplus M_5(\mathbb{Q}) \oplus M_4(\mathbb{Q}) \oplus \mathbb{Q}.$$

For each partition λ we construct a block matrix X_λ as described in the previous section. It has size $14d_\lambda \times 42d_\lambda$; the rows of blocks are labeled by the basic nonassociative monomials of degree 5, and the columns of blocks are labeled by the normal D -types of degree 5. Thus each row of blocks contains the representation

matrices of the terms of the expansion of the corresponding basic monomial into the free dendriform algebra. We compute $\text{RCF}(X_\lambda^t)$, find the rank and the nullity of X_λ^t , extract a basis for the nullspace of X_λ^t , put the basis vectors into the rows of another matrix N_λ , and compute $\text{RCF}(N_\lambda)$. The nonzero rows of $\text{RCF}(N_\lambda)$ give a canonical set of generators for the S_5 -module of identities corresponding to partition λ . See the 4 columns labelled “all” in Table 5.

We need to check that these identities are consequences of the pre-Jordan identities. Thus we compare them with the liftings to degree 5 of the defining identities for pre-Jordan algebras. For each partition λ of 5 we construct a $12d_\lambda \times 14d_\lambda$ matrix L_λ . The rows of blocks correspond to the 12 liftings (6) of the defining identities for pre-Jordan algebras. The columns of blocks are labeled by the normal D -types of degree 5. For each $i = 1, \dots, 12$ and $j = 1, \dots, 14$ we collect the terms in the i -th lifting which have the j -th normal D -type, and store the corresponding representation matrix in the block in position (i, j) . We compute $\text{RCF}(L_\lambda)$; the nonzero rows of $\text{RCF}(L_\lambda)$ give a canonical set of generators for the S_5 -module of lifted identities corresponding to partition λ . See the 3 columns labelled “lifted” in Table 5.

For every partition λ , the rank of the lifted identities equals the nullity of all identities; moreover, $\text{RCF}(N_\lambda) = \text{RCF}(L_\lambda)$, omitting zero rows. We conclude that there are no special identities in degree 5 for the pre-Jordan product in the free dendriform algebra.

Before proceeding to degree 6, we make some comments. If there are special identities in some degree $n \geq 5$, then there will also be special identities in every degree $> n$. To make this inference, we need to assume that the domain of the expansion map is the free unital nonassociative algebra. If $f(x_1, \dots, x_n)$ is a special identity in degree n , then in the lifting $f(x_1 \cdot x_{n+1}, \dots, x_n)$ we can set $x_{n+1} = 1$, showing that if the lifting is not special then neither is the original identity. Thus, if we can prove that there are no special identities in degree 7, it follows that there are also no special identities in degrees 5 or 6. However, for completeness, we include the results for degree 6.

Degree 6. For degree 6 we proceed analogously to the degree 5 case. There are 12 lifted pre-Jordan identities in degree 5, each of which produces 7 liftings to degree 6, for a total of 84 lifted identities in degree 6. In degree 6, there are 42 binary association types, and 132 normal D -types. Hence for each partition λ of 6, with corresponding dimension d_λ , the matrix L_λ of lifted identities has size $84d_\lambda \times 42d_\lambda$, and the expansion matrix X_λ^t has size $132d_\lambda \times 42d_\lambda$. In every case, we find that the rank of L_λ equals the nullity of X_λ^t , and so there are no new identities in degree 6; see Table 6.

At this point, the number of lifted identities is increasing rapidly, and so the matrix L_λ is becoming very large. We can reduce the amount of memory required as follows. Suppose that there are k lifted identities in degree n . Write c_n for the number of binary association types in degree n . Normally, the matrix L_λ would have size $kd_\lambda \times c_n d_\lambda$. However, if $k > c_n$, we can take L_λ to have size $(c_n + 1)d_\lambda \times c_n d_\lambda$. We then process the lifted identities one at a time, storing their representation matrices in the last row of $d_\lambda \times d_\lambda$ blocks, and compute the RCF of the matrix after each lifted identity. This saves memory but uses more time since we must perform many more row reductions.

#	λ	d_λ	lifted (L_λ)			all (X_λ^t)				new
			rows	cols	rank	rows	cols	rank	nullity	
1	5	1	12	14	7	42	14	7	7	0
2	41	4	48	56	21	168	56	35	21	0
3	32	5	60	70	20	210	70	50	20	0
4	311	6	72	84	22	252	84	62	22	0
5	221	5	60	70	14	210	70	56	14	0
6	2111	4	48	56	9	168	56	47	9	0
7	11111	1	12	14	1	42	14	13	1	0

TABLE 5. Degree 5: matrix ranks for all representations

#	λ	d_λ	lifted (L_λ)			all (X_λ^t)				new
			rows	cols	rank	rows	cols	rank	nullity	
1	6	1	84	42	27	132	42	15	27	0
2	51	5	420	210	110	660	210	100	110	0
3	42	9	756	378	170	1188	378	208	170	0
4	411	10	840	420	176	1320	420	244	176	0
5	33	5	420	210	87	660	210	123	87	0
6	321	16	1344	672	247	2112	672	425	247	0
7	3111	10	840	420	138	1320	420	282	138	0
8	222	5	420	210	67	660	210	143	67	0
9	2211	9	756	378	112	1188	378	266	112	0
10	21111	5	420	210	53	660	210	157	53	0
11	111111	1	84	42	8	132	42	34	8	0

TABLE 6. Degree 6: matrix ranks for all representations

#	λ	d_λ	lifted (L_λ)			all (X_λ^t)				new
			rows	cols	rank	rows	cols	rank	nullity	
1	7	1	133	132	95	429	132	37	95	0
2	61	6	798	792	504	2574	792	288	504	0
3	52	14	1862	1848	1060	6006	1848	788	1060	0
4	511	15	1995	1980	1099	6435	1980	881	1099	0
5	43	14	1862	1848	992	6006	1848	856	992	0
6	421	35	4655	4620	2333	15015	4620	2287	2333	0
7	4111	20	2660	2640	1259	8580	2640	1381	1259	0
8	331	21	2793	2772	1333	9009	2772	1439	1333	0
9	322	21	2793	2772	1269	9009	2772	1503	1269	0
10	3211	35	4655	4620	2035	15015	4620	2585	2035	0
11	31111	15	1995	1980	800	6435	1980	1180	800	0
12	2221	14	1862	1848	751	6006	1848	1097	751	0
13	22111	14	1862	1848	705	6006	1848	1143	705	0
14	211111	6	798	792	269	2574	792	523	269	0
15	1111111	1	133	132	38	429	132	94	38	0

TABLE 7. Degree 7: matrix ranks for all representations

Degree 7. There are 84 lifted pre-Jordan identities in degree 6, each of which produces 8 liftings to degree 7, for a total of 672 lifted identities in degree 7. In degree 7, there are 132 binary association types. Hence we take L_λ to have size $133d_\lambda \times 132d_\lambda$. In degree 7, there are 429 normal D -types, and so the expansion matrix X_λ^t has size $429d_\lambda \times 132d_\lambda$. In every case, we find that the rank of L_λ equals the nullity of X_λ^t , and so there are no new identities in degree 7; see Table 7.

For degree 7, we kept track of which lifted identities increased the rank of L_λ for at least one partition λ . This reduced the number of pre-Jordan identities in degree 7 from 672 to 133. Each of these has 9 liftings to degree 8, for a total of 1197 (much smaller than $9 \cdot 672 = 6048$). In this way, we substantially speed up the computations in degree 8.

7. SPECIAL IDENTITIES IN DEGREE 8

Owing to the size of the representation matrices in degree 8, and limits on available memory, we had to split the problem into smaller parts and process each representation in a separate Maple worksheet; this takes more CPU time but requires less memory. In degree 8 we find new (and hence special) identities in partitions $\lambda = 431, 422, 332, 3311$ and 3221 , see Table 8.

#	λ	d_λ	lifted (L_λ)			all (X_λ^t)				new
			rows	cols	rank	rows	cols	rank	nullity	
1	8	1	479	429	339	479	429	90	339	0
2	71	7	3353	3003	2174	3353	3003	829	2174	0
3	62	20	9580	8580	5778	9580	8580	2802	5778	0
4	611	21	10059	9009	5939	10059	9009	3070	5939	0
5	53	28	13412	12012	7671	13412	12012	4341	7671	0
6	521	64	30656	27456	16930	30656	27456	10526	16930	0
7	5111	35	16765	15015	8951	16765	15015	6064	8951	0
8	44	14	6706	6006	3728	6706	6006	2278	3728	0
9	431	70	33530	30030	17721	33530	30030	12308	17722	1
10	422	56	26824	24024	13812	26824	24024	10211	13813	1
11	4211	90	43110	38610	21676	43110	38610	16934	21676	0
12	41111	35	16765	15015	8032	16765	15015	6983	8032	0
13	332	42	20118	18018	10039	20118	18018	7977	10041	2
14	3311	56	26824	24024	13056	26824	24024	10967	13057	1
15	3221	70	33530	30030	15853	33530	30030	14176	15854	1
16	32111	64	30656	27456	13956	30656	27456	13500	13956	0
17	311111	21	10059	9009	4289	10059	9009	4720	4289	0
18	2222	14	6706	6006	2978	6706	6006	3028	2978	0
19	22211	28	13412	12012	5803	13412	12012	6209	5803	0
20	221111	20	9580	8580	3929	9580	8580	4651	3929	0
21	2111111	7	3353	3003	1262	3353	3003	1741	1262	0
22	11111111	1	479	429	158	479	429	271	158	0

TABLE 8. Degree 8: matrix ranks for all representations

There are 429 binary association types in degree 8, and 1430 normal D -types. To save memory, we processed the lifted identities 50 at a time, so that L_λ has size $(429 + 50)d_\lambda \times 429d_\lambda$. We also used a similar strategy for the expansion matrix X_λ^t . For each basic nonassociative monomial, we compute the expansion using

the pre-Jordan product and replace each term by its normalized form (as a linear combination of normal D -words). We then split each expansion by considering the normal D -types 50 at a time; in each iteration, we process those terms in the normalized expansion whose normal D -types are among the current 50. Hence X_λ^t has size $(429 + 50)d_\lambda \times 429d_\lambda$, the same as L_λ . At the end of this computation, the ranks will be the same as if we had processed the expansions all at once.

We do not give explicit expressions for the new special identities, since they involve far too many terms.

The column labelled ‘new’ in Table 8 is identical to the corresponding column in [12, Table 7] which gives the new identities in degree 8 for the Jordan diproduct in an associative dialgebra. A comparison of the results of this paper with those of [12] shows that for $4 \leq n \leq 8$, the multiplicity of the irreducible S_n -module $[\lambda]$ in the space of new identities is the same for the pre-Jordan product in the free dendriform algebra and the Jordan diproduct in the free associative dialgebra. This result does not hold for $n = 3$, since the pre-Jordan product satisfies no identities, but the Jordan diproduct satisfies right commutativity. In the next section we state a conjecture which explains this fact in terms of Koszul duality of operads.

8. CONJECTURES

Our first conjecture extends the results of this paper and [12] to degree n .

Conjecture 8.1. *Let Dias and $\text{Dend} = \text{Dias}^!$ be the dual operads of associative dialgebras and dendriform algebras respectively.*

In degree $n \leq 3$, every multilinear polynomial identity satisfied by the Jordan diproduct in the free Dias -algebra is a consequence of right commutativity, $x(yz) - x(z y) \equiv 0$. Let P_n be the S_n -module of multilinear right-commutative polynomials in degree n (the multilinear subspace in degree n of the free right commutative algebra), let $Q_n \subseteq P_n$ be the submodule of identities satisfied by the Jordan diproduct in the free right-commutative algebra, and let $R_n \subseteq Q_n$ be the submodule of identities which are consequences of identities in lower degrees.

In degree $n \leq 3$, there are no polynomial identities satisfied by the pre-Jordan product in the free Dend -algebra. Let P'_n be the S_n -module of multilinear nonassociative polynomials in degree n (the multilinear subspace in degree n of the free nonassociative algebra), let $Q'_n \subseteq P'_n$ be the submodule of identities satisfied by the pre-Jordan product in the free dendriform algebra, and let $R'_n \subseteq Q'_n$ be the submodule of identities which are consequences of identities in lower degrees.

Then for all $n \geq 4$ we have an isomorphism of S_n -modules $Q_n/R_n \cong Q'_n/R'_n$.

This conjecture suggests that a similar isomorphism might hold more generally between the polynomial identities for corresponding bilinear operations in the free algebras over any dual pair of binary quadratic non- Σ operads. (We recall from Loday [23] that a non- Σ operad \mathbf{O} is defined by the conditions that the operations have no symmetry and the relations contain only the identity permutation of the variables; this implies that each space $\mathbf{O}(n)$ is a direct sum of copies of the regular representation of S_n .) The duality theory for quadratic operads is well-developed, but we need to clarify the meaning of the phrase ‘corresponding bilinear operations’.

The space of operations $\text{Dias}(2)$ is the tensor product of the vector space V with basis $\{\dashv, \vdash\}$ with the group algebra FS_2 ; thus $\text{Dias}(2)$ has basis

$$x \dashv y, \quad y \dashv x, \quad x \vdash y, \quad y \vdash x.$$

Similarly, the space of operations $\text{Dend}(2)$ is the tensor product of the vector space W with basis $\{\prec, \succ\}$ with the group algebra FS_2 ; thus $\text{Dend}(2)$ has basis

$$x \prec y, \quad y \prec x, \quad x \succ y, \quad y \succ x.$$

By the duality between Dias and Dend , we know that $\text{Dend}(2) = \text{Dias}(2)^* \otimes_{FS_2} (\text{sgn})$ where (sgn) is the signature representation.

The Jordan diproduct in $\text{Dias}(2) = V \otimes FS_2$ is the element $x \dashv y + y \vdash x$. The annihilator of this element in $\text{Dias}(2)^*$ is $(x \dashv y)^* - (y \vdash x)^*$; recall that S_2 acts only on the second factor in the tensor product. The corresponding element in $\text{Dend}(2)$ requires twisting by the signs of the permutations, giving $(x \dashv y)^* + (y \vdash x)^*$, and this represents the pre-Jordan product $x \succ y + y \prec x$. (We identify \dashv with \succ and \vdash with \prec , opposite to Loday [23, Proposition 8.3].) In other words, we can identify the spaces of operations in this dual pair of operads in a natural way so that the pre-Jordan product corresponds to the Jordan diproduct. For further details, see [23, Proposition B.3].

Now consider a variety V of algebras with k binary operations, and assume that the operations have no symmetry. The space E of generating operations is isomorphic as an S_2 -module to the direct sum of k copies of FS_2 . Therefore both E^* and $E^\vee = E^* \otimes (\text{sgn})$ are canonically isomorphic to E . This means that for any bilinear operation M in the variety V (that is, any element M of E), there is a canonical corresponding bilinear operation M^\vee in any variety V^\vee defined by the operations E^\vee . In particular, we consider the case where the variety V is defined by a quadratic operad, and the variety V^\vee is defined by the dual operad.

Conjecture 8.2. *Let \mathcal{O} and $\mathcal{O}^!$ be a dual pair of binary quadratic non- Σ operads. Let \circ be a bilinear operation in $\mathcal{O}(2)$, and let \circ' be the corresponding operation in $\mathcal{O}^!(2)$. Let \mathcal{V} be the variety defined by the multilinear polynomial identities of degree ≤ 3 satisfied by the operation \circ in the free \mathcal{O} -algebra, and let \mathcal{V}' be the variety defined by the multilinear polynomial identities of degree ≤ 3 satisfied by the operation \circ' in the free $\mathcal{O}^!$ -algebra.*

Let P_n be the S_n -module of multilinear \mathcal{V} -polynomials in degree n (the multilinear subspace in degree n of the free \mathcal{V} -algebra), let $Q_n \subseteq P_n$ be the submodule of identities satisfied by the operation \circ in the free \mathcal{O} -algebra, and let $R_n \subseteq Q_n$ be the submodule of identities which are consequences of identities in lower degrees.

Let P'_n be the S_n -module of multilinear \mathcal{V}' -polynomials in degree n (the multilinear subspace in degree n of the free \mathcal{V}' -algebra), let $Q'_n \subseteq P'_n$ be the submodule of identities satisfied by the operation \circ' in the free $\mathcal{O}^!$ -algebra, and let $R'_n \subseteq Q'_n$ be the submodule of identities which are consequences of identities in lower degrees.

Then for all $n \geq 4$ we have an isomorphism of S_n -modules $Q_n/R_n \cong Q'_n/R'_n$.

A further generalization of this conjecture to operads which are not necessarily binary might be possible, but we will not pursue this question here.

9. OPEN PROBLEMS

Dendriform algebras can be obtained from associative algebras by splitting the operation, and in the same way quadri-algebras can be obtained from dendriform algebras. This splitting procedure can be repeated any number of times, leading to a sequence of structures with 2^n binary nonassociative operations, satisfying a coherent set of polynomial identities in degree 3, which imply that the sum of the

operations is associative. These structures are often called Loday algebras [27] or ABQR algebras [16]; see also [2].

As illustrated in Table 1, Lie and Jordan algebras can be obtained from bilinear operations on associative algebras, and analogously, pre-Lie and pre-Jordan algebras can be obtained from bilinear operations on dendriform algebras. This process of obtaining nonassociative structures from associative structures can be extended to obtain L -dendriform algebras [4] and J -dendriform algebras [3] from quadri-algebras.

An important open problem is to extend the Gröbner-Shirshov basis for free dendriform algebras [14] to quadri-algebras and beyond. This would allow us to extend the methods of the present paper to polynomial identities for J -dendriform algebras. It would also be very useful to have a theory of Gröbner bases for arbitrary ideals in the free dendriform algebra. In the dual setting of free associative dialgebras, this has been accomplished by Bokut et al. [6]. This development is a necessary first step for the construction of universal enveloping dialgebras (resp. dendriform algebras) for Leibniz algebras and Jordan dialgebras (resp. pre-Lie and pre-Jordan algebras).

Dual to quadri-algebras is the variety introduced by Vallette [27, §5.6]. This variety can also be obtained by applying the KP algorithm [9, 20] to associative dialgebras. It is an open problem to study analogues of Lie and Jordan structures in this setting. This also raises the question of finding an analogue of the KP algorithm in the dendriform setting: this algorithm takes a variety of algebras and produces the corresponding variety of dialgebras; the analogous procedure in the dendriform setting would involve splitting of the operations to produce the corresponding variety of dendriform algebras. The BSO algorithm [9, 20] takes a multilinear n -ary operation in an algebra and produces the corresponding operations in a dialgebra; it would be useful to have an analogue of this process for dendriform algebras.

It would be natural to consider analogues of other nonassociative structures in the settings of dendriform algebras, quadri-algebras, and the algebras defined by the dual operads. For alternative structures, see [22, 26]. The same questions can be asked in the context of the associative trialgebras and dendriform trialgebras introduced by Loday and Ronco [24].

As a final note, we recall that the operads defining Jordan algebras, Jordan dialgebras, and pre-Jordan algebras, are all binary and cubic. An important problem is to extend to cubic operads the theory of Koszul duality [17].

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